Optimal targeting strategy in a network under positive externalities

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COST-Comsoc Istanbul November 2015

- A planner (e.g. firm, government, health authority) aims to enhance agents' activity
- Social network under positive externalities
- Tool: targeting of nodes/agents by allocating a fixed amount of 'resources'
- Examples: viral marketing, control of contagion, criminal activity ...

Questions and objectives

- How is the planner's amount optimally allocated? Is it concentrated on few agents or dispersed among numerous ones?
- What is the value of information on the interaction structure?
- So far mostly two models: linear models of interactions or 0-1 model.
- Here: Individual's action is a continuous variable
  Tractable non linear model to study equilibria (steady states) and planner's impact

Some insights

- Planner's strategy relies on
  - individuals' impact totals (out-degrees) and Centrality Katz-Bonacich indices under linear interaction
  - other network's characteristic in interaction with diminishing returns, 'attention' and not only impact matters, structure of joint impact
  - stronger properties when impact totals are equal
- The value of information is almost always positive, and is linked to the heterogeneity in the network

1. Equilibrium in an interaction model
2. Constant returns to exposure
   - Centrality Katz-Bonacich indices
3. Diminishing returns to exposure
   - Joint impact
   - General case: some results.
4. Concluding remarks

Examples

- linear response
  - strategic games with quadratic payoffs $\rightarrow$ a linear 'best' reply
    Ballester, Calvo Armengol, Zenou [2006]
    action= criminal activity, effort ...
    objective: suppress the 'key player', i.e. a node
  - pricing model with discrimination of the nodes (Bloch and Querou [2013], Candogan, Bimpikis, Ozdaglar [2012]) Fainmesser and Galeotti [2013])
    action= probability of purchase or adoption
    profit objective
  - financial network : Demange [2015]
    action= proportion of debt repayments (lower and upper bound)
    objective: inject cash into banks to maximize overall repayments

Binary variables/Threshold models

- adoption/contagion process: 0-1 model
  threshold models or SIR model
  Schelling [1969], Morris [2003], Domingos and Richardson [2001] in a marketing context, Dodds and Watts [2004] in biology
- planner's strategy: choose a subset to initiate the maximal diffusion
- statistical insights
- computational issues in 0-1 threshold models: Kempe, Kleinberg and Tardos [2003]

Impact and exposures

- n agents, take actions, $\theta_i \geq 0$ for agent i, $\boldsymbol{\theta} = (\theta_i)$
- Bilateral Impacts:
  $\pi_{ij} \geq 0 =$ *impact* of i on j or j's *attention* to i $\pi_{ii} = 0$
  example: network with π_{ij} equal to 0 or 1
- Exposures :
  Given $\boldsymbol{\theta} = (\theta_i)$, $\tau_j(\boldsymbol{\theta}) = \sum_i \pi_{ij}\theta_i$ is the (total) exposure of j.
- Reaction to exposures : determined by a *response* function f

Reaction

- Reaction to exposure:

$$\begin{aligned}\theta_i &= z_i + f(\sum_j \pi_{ji}\theta_j) \quad \text{if } \geq 0 \\ &= 0 \quad \text{otherwise}\end{aligned}$$

f continuous from R_+ to R_+, $f(0) = 0$.

An equilibrium: $\boldsymbol{\theta} = (\theta_i)$ for which each θ_i is the reaction to i's exposure.

- $z_i = x_i + y$
  x_i: planner's allocation to i (to be determined)
  y: i's action level in isolation

Equilibrium under strategic complements

- Assume f is increasing: actions are strategic complements.
- Equilibria are 'well behaved' and easy to find (iterate reactions) Topkis [1979]
  $\rho(\boldsymbol{\pi})=$ dominant eigenvalue of $\boldsymbol{\pi}$.
- *Assumption L(ipshitz)*: $f'(\tau)\rho(\boldsymbol{\pi}) < 1$ for all τ

Under assumption L, an equilibrium exists and is unique.

Can be relaxed, but not uniqueness
I consider decreasing, constant, or increasing returns to exposure : f concave, linear or convex

The planner's objective

- 'Planner' aims at improving aggregate activity : $\sum_i \theta_i$
  Endowed with amount $m \geq 0$ to distribute.
- a targeting strategy n-vector $\boldsymbol{x} = (x_i)$, x_i changes y into $z_i = y + x_i$, hence changes equilibrium actions
- Ex: x_i : discount or charge
  x_i cash, time spent 'positive' case: each x_i must be ≥ 0
- budget constraint:
  – positive setting: $\boldsymbol{x} = (x_i), \sum_i x_i = m$
  – unconstrained setting: extracted amount is limited by $y + f(\tau_i)$

Optimal strategies

$\boldsymbol{x}$ is optimal if it maximizes equilibrium aggregate activity $\sum_i \theta_i$ over all feasible strategies.

- The planner accounts for the full impact of externalities

Linear model: $f(\tau) = \delta\tau$

Unconstrained setting. Let $\pi_+^{\max} = \max_j \pi_{i+}$

If $\delta\pi_+^{\max} \geq 1$, then aggregate action can be made infinitely large.

If $\delta\pi_+^{\max} < 1$, then an optimal strategy exists, targets the nodes with maximal impact total and 'exploits' others, i;e. leave them with null action.

Positive setting. Let $\boldsymbol{\mu} = (\mathbb{I} - \delta\boldsymbol{\pi})^{-1}\mathbb{1}$ 'multipliers'

Optimal positive strategies: the feasible ones that target individuals with maximum multiplier $\mu^{max} = \max_j \mu_j$.

Linear model: Implications Positive strategies

- Multiplier : centrality index in the impact network (Katz [1953] Bonacich [1987]) :

$$\boldsymbol{\mu} = (I - \delta\boldsymbol{\pi})^{-1}\mathbb{1} = \mathbb{1} + \delta\boldsymbol{\pi}\mathbb{1} + \delta^2\boldsymbol{\pi}^2\mathbb{1} \cdots + \cdots$$

  $\mu_i =$ number of discounted paths from i in the impact network.
- Actions and multipliers are 'dual' to each other

  Actions= linear in the centrality index in the attention network
- In a non symmetric network

  Targets are not necessarily the individuals with the largest action

Linear model: Value of information.

Benchmark: A uniform (or a random) strategy allocates equal amount to each $\boldsymbol{x} = \frac{m}{n}\mathbb{1}$,

Benefit from the optimal strategy over the uniform one:

$$[\frac{1}{(1-\delta\pi_{+}^{\max})} - (\frac{1}{n}\sum_{i}\mu_i)]m \text{ unconstrained case}$$

$$[\mu^{max} - (\frac{1}{n}\sum_{i}\mu_i)]m \text{ positive case}$$

- Value reflects the heterogeneity in the impact matrix.

Equal impact

- Null information values only when impact totals are equal :

$$\sum_j \pi_{ij} \text{ identical across } i$$

- ex: i delivers a speech to each of his followers separately;
  π_{ij} =the proportion of time devoted by i to each
  θ_i = the overall time i allocates to the action.

Diminishing returns to exposure

- no explicit solution for the equilibria and strategies
- one can exploit the geometry of equilibria due to complementarity: the set of actions $\boldsymbol{\theta} \geq \mathbf{0}$ that satisfy

$$\theta_i \leq z_i + f(\sum_j \pi_{ji}\theta_j) \text{ for each } i$$

  has a greatest element, which is the equilibrium associated to $\boldsymbol{z}$,
- put the planner's problem as a concave program.
- optimal strategies are characterized by 'multipliers' in the positive case
- Here: consider the unconstrained quadratic case.

Quadratic unconstrained case

$$f(\tau) = \delta\tau - \frac{\gamma}{2}\tau^2 \text{ for } \tau \leq \frac{\delta}{\gamma} \text{ constant thereafter}$$

- Define i, j-*joint impact* by $\sigma_{ij} = \sum_k \pi_{ik}\pi_{jk}$. $\boldsymbol{\sigma} = \boldsymbol{\pi}\widetilde{\boldsymbol{\pi}}$.

  congruence in i and j impact
  In a network: $\sigma_{ij} =$ number of nodes impacted by both i and j
- Given $\boldsymbol{\theta}$, call $\sum_j \sigma_{ij}\theta_j$ i's weighted joint impact at $\boldsymbol{\theta}$
- To simplify: m small, $\boldsymbol{\pi}$ invertible

At the optimal strategy:

$$\delta\pi_{i+} - \gamma \sum_{j \in I} \sigma_{ij}\theta_j \text{ is maximum for } i \text{ with positive action}$$

Strategy adjusted to actions $x_i = \theta_i - f(\tau_i)$.
Extract the maximum from those with null actions

- Strategy trades-off between
  1. targeting the agents whose impact is maximal and
  2. targeting those who have a small joint impact, i.e. who have an impact on agents difficult to influence, with little attention

Quadratic response function-equal impact

- No trade-off in the case of equal impact.
- actions are proportional to the unit vector $\widehat{\boldsymbol{\theta}}$ that minimize $\widetilde{\boldsymbol{\theta}}\boldsymbol{\sigma}\boldsymbol{\theta}$, i.e. the variation in exposure levels $\sum_i \tau_i^2$
- Full support only if exposure levels can be equalized, $\widetilde{\boldsymbol{\pi}}\widehat{\boldsymbol{\theta}} = \mathbb{1}$ Equivalent to : no subset of nodes is 'attention-dominated'
- Extends to any f: If exposure levels can be equalized, $\widetilde{\boldsymbol{\pi}}\widehat{\boldsymbol{\theta}} = \mathbb{1}$ always optimal to induce them

General f

- Take second-order approximation of f
- Simulate networks according to Erdos Renyi and adjust the rows
  The support of $\widehat{\boldsymbol{\theta}}$ is almost never N: positive value of information.
- $\neq$ in the linear case, where every strategy is optimal

- The optimal strategies and actions are positive whatever concave f and resources m iff both impact and attention totals are equal. In that case the uniform strategy is optimal.
- The targeting strategy is not necessarily monotone in m. Root of difficulties in computing targeting strategies

Large resources

- For large m, the optimal strategies depend on the limit to marginal exposure $\omega = \lim_{\tau \to \infty} f'(\tau)$
- If $\omega > 0$, the interaction becomes close to linear (provided all exposures become large)
- Even for a null ω, the benefit from the knowledge of the network is positive under most circumstances

Concluding remarks

- Linear model of interaction leads to quite specific targeting strategies
- Under diminishing returns to exposure, the differences in attention totals and the 'joint' impact matter
- Value of information almost always positive, related to the heterogeneity in the network

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